# How often does the oldest person alive die?
# A demographic application of queueing theory.

Extended Abstract for Submission to the 2018 Annual Meeting of the PAA

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**Abstract**

We develop a formal model to answer the question: How often does it happen that the oldest person alive dies? We assume that the oldest person alive is at least 110 years old and that the number of people turning 110 is constant over time. Previous research has shown that mortality reaches a plateau at those advanced ages. According to standard queueing theory, the population above 110 years in our model follows a Poisson distribution with parameter $\lambda/\mu$ where $\lambda$ depicts the rate at which people turn 110 and $\mu$ represents the force of mortality. These analytical results were consistent with the results from our simulation studies. Our simulation studies also suggested that the waiting time between the deaths of the (respective) oldest person alive follows an exponential distribution with parameter $\mu$ and, thus, with a mean duration of $1/\mu$ if $\lambda > \mu$. At the moment, we do not have an expression for the case of $\lambda \leq \mu$.

## 1 Introduction

How often does it happen that the oldest person alive dies? We became interested in this question when we read that Emma Morano died on 15 April 2017, aged 117 years and 116 days (e.g., BBC, 2017). She was the last person alive who was born in the 19th century and, consequently, the oldest person alive. We had the impression the event of the death of the oldest person alive occurs fairly regularly, about once a year or every other year. A search on the BBC website resulted in the pattern depicted in the upper panel of Figure 1. During the span of about 15 years, the respective oldest person alive died 18 times; 17 times it was a woman and once a man. The lower panel shows an estimate of the density of the duration between these events. It shows a large concentration throughout for the duration of up to one year and declines steadily thereafter. Since the lower panel is based on only 17 time differences, one

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should be cautious interpreting the density.

Our goal as formal demographers is to find a suitable model to analyze how often the oldest person alive dies. Or, described differently, what is the mean time we have to wait until we can expect to see these newspaper reports again?

## 2 Method: Queueing Theory

Under a few (justifiable) simplifications, queueing theory provides an appropriate framework for our analysis. As this theory is rarely employed by demographers, we are a bit more verbose than an expert in the field would expect.

Following the canonical notation introduced by Kendall in 1953, stochastic processes in queueing theory are characterized by three components:

1. The first component is called the *input process* (Bhat, 2015). In standard queueing theory, it refers to the number of people arriving in a queue. Since all deaths shown in Figure 1 occur among people aged 114 or older, we define our input process as the number of people per time unit who become supercentenarians, i.e. individuals who celebrate their 110th birthday. For simplicity, we assume that the number of arrivals is Poisson distributed with a constant intensity over time $\lambda$. As the number of arrivals at time point $t$ is independent of the number of arrivals at $t-1$, Kendall denoted this Markovian process with an $M$.

2. The second component is called the *serving mechanism* (Bhat, 2015). In standard queueing theory it refers to the rate at which customers are served. This corresponds to the force of mortality in our application. Since the oldest person alive is at least 110 years, we also make a simple assumption for the service mechanism: Jutta Gampe (2010), among others, has shown that mortality reaches a plateau at age 110. Consequently, we expect the instantaneous intensity of the service, namely death, to be constant as well, so that the time to the next service or death obeys an exponential distribution with hazard $\mu$. As this distribution is "memoryless", it is also represented in Kendall's notation as an $M$ process.

3. The third component indicates the number of servers: If we use the example of a queue at the check-in at an airport, this third component denotes the number of open counters. In our application, we have an infinite number of servers as nobody has to wait for another person to die first.

In summary, we model the occurrence of deaths among individuals with completed age of 110 or more as a $M/M/\infty$ queue. The properties of $M/M/\infty$ queues are well understood (e.g., Bhat, 2015; Giambene, 2014; Gross et al., 1998):

- There is a steady state solution in a $M/M/\infty$ system, i.e. a stationary distribution of people aged 110+ exists.
- This distribution is Poisson $(\lambda/\mu)$.
- Consequently, the probability that exactly $n$ people aged 110+ are alive is:

$$P(N = n) = \frac{\left(\frac{\lambda}{\mu}\right)^n e^{-\frac{\lambda}{\mu}}}{n!}$$

- The mean number of people aged 110+ is therefore $\frac{\lambda}{\mu}$.
- Since deaths are exponentially distributed with rate parameter $\mu$, life expectancy of each individual is $1/\mu$.

What is not treated in standard text books is the time between deaths of the oldest person alive.

## 3 Method: Simulation

To answer this question, we conducted simulation studies. Using days as our unit of time and varying values for $\lambda$ and for $\mu$ we ran simulations of 50,000 days.

As an example of our simulations, Figure 2 depicts the density of the simulated population sizes as a histogram. The expected theoretical density is shown by a solid green line. The two vertical dotted reference lines denote the empirically measured mean (red) as well as the theoretical mean of the distribution (blue). We can see that the analytical description of a Poisson distribution with a parameter of $\lambda/\mu = 10/0.5 = 20$ holds.

As expected from the theory, the population size is also stationary as illustrated in Figure 3. Gross et al. (1998), for instance, state about the Poisson distribution with parameter $\lambda/\mu$ (p. 81): "Note that the value of $\lambda/\mu$ is not in any way restricted for the existence of a steady state solution."

If $\lambda > \mu$, the waiting times in our simulation between our events of interest followed an exponential distribution with parameter $\mu$. Consequently, the mean waiting period was $1/\mu$. This is illustrated in Figure 4.

Those results did not apply when $\mu > \lambda$. Although we still obtain an exponential distribution, the "expected" theoretical fit of an exponential distribution is clearly not fitting the data in Figure 5. Instead of a value of $\mu = 5$ as we would have assumed from the previous figure, the maximum likelihood estimate for an exponential distribution from the data resulted in a value of $\hat{\mu} = 1.645$. Consequently, the main waiting time was not $1/5 = 0.2$ but $1/1.645 = 0.608$.

Future steps will include more detailed analysis for this case of $\mu > \lambda$. We might also relax the assumption of a constant $\lambda$.

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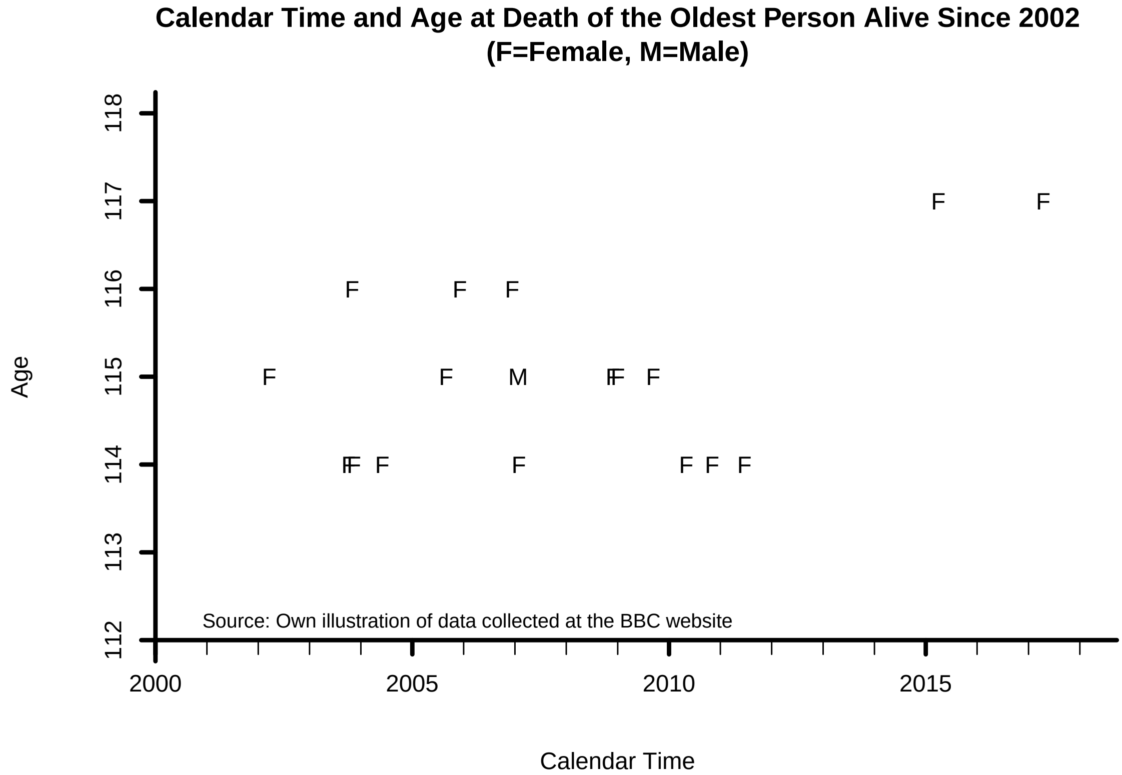


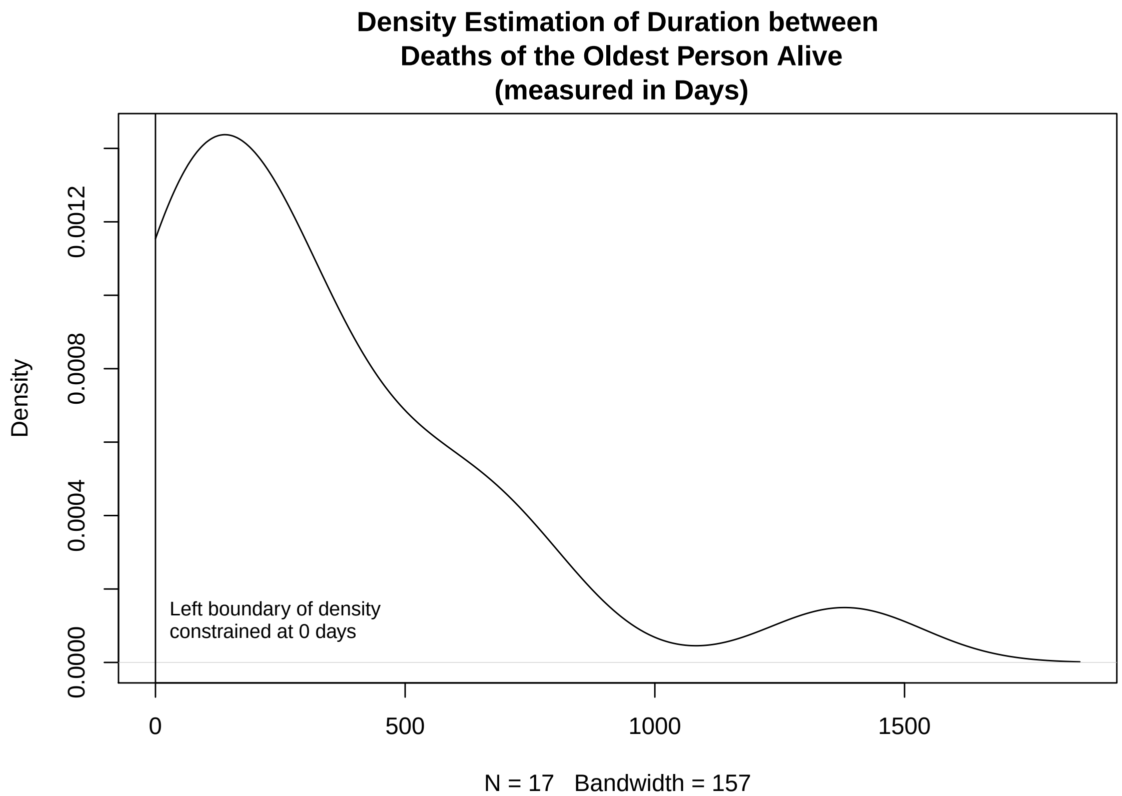


Figure 1: Upper panel: Calendar time and age at death of the oldest person alive since 2002. Lower panel: Density estimation of duration between deaths of the oldest person alive. default kernel density estimation in R using the differences between consecutive time points from the upper panel as input data. The density has been constrained to start at 0.

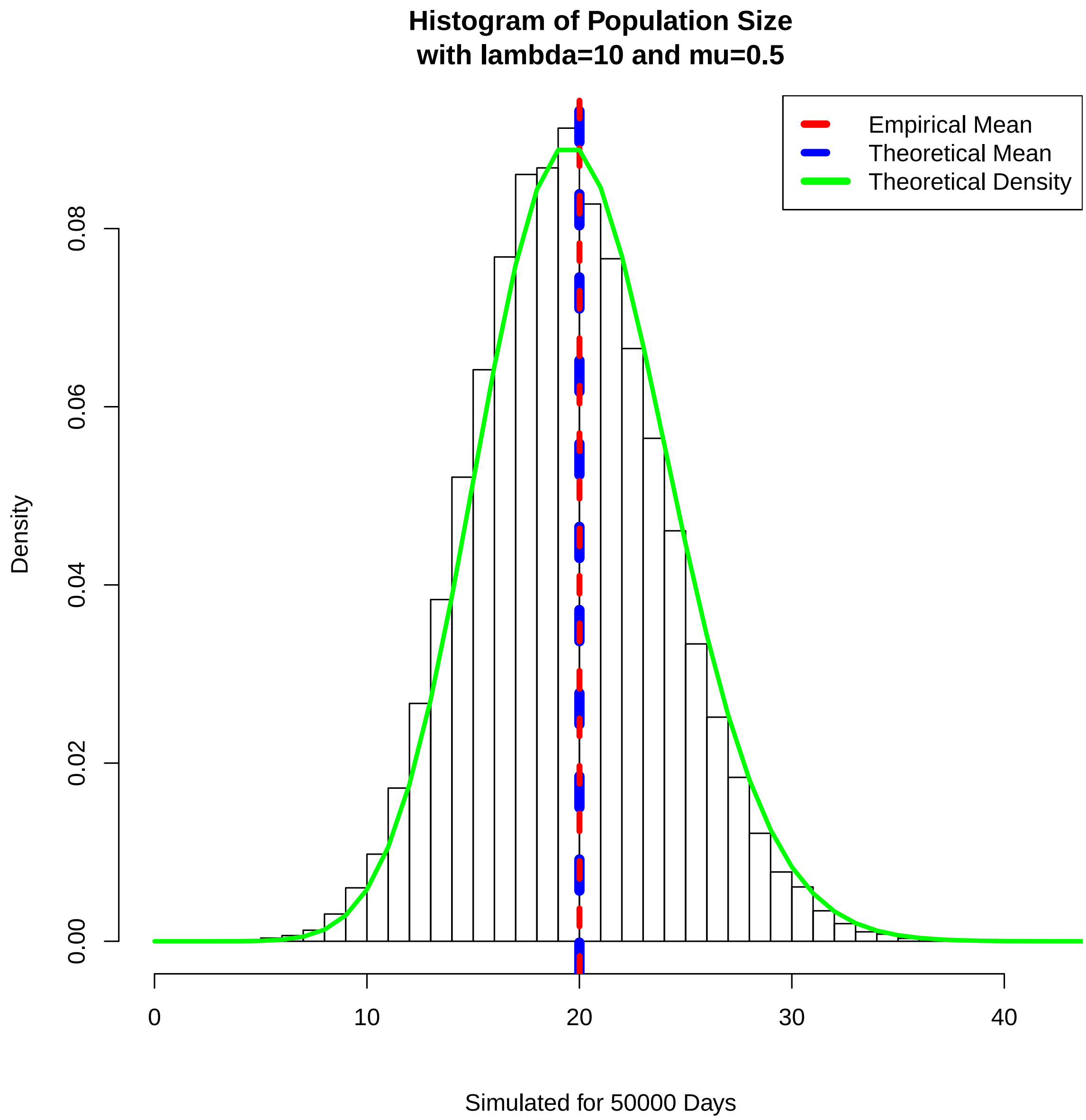


Figure 2: The density of the simulated population sizes is depicted as a histogram. The expected theoretical density is shown by a solid green line. The two vertical dotted reference lines denote the empirically measured mean (red) as well as the theoretical mean of the distribution (blue). The simulated data are based on $\lambda = 10$ and $\mu = 0.5$.

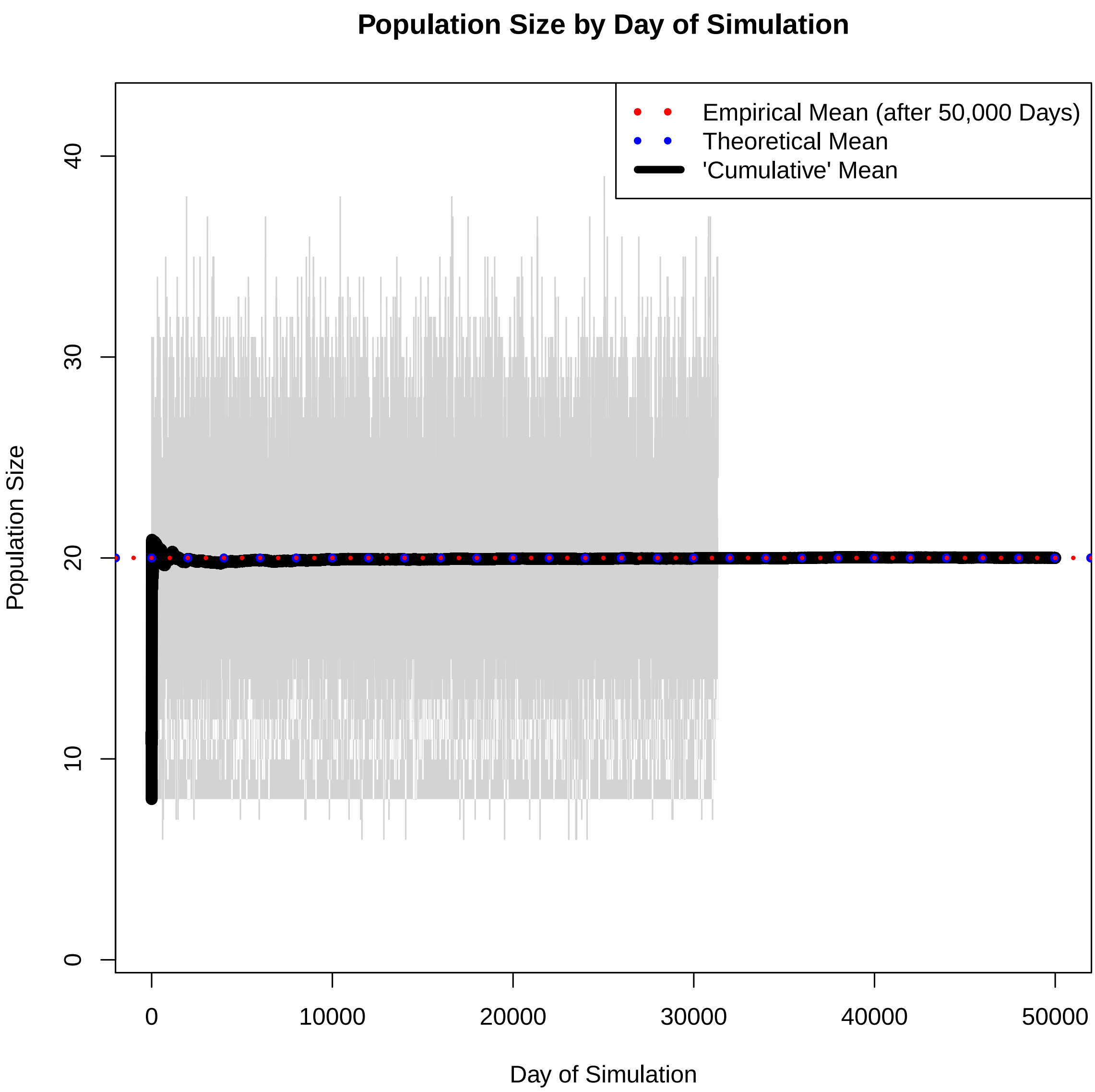


Figure 3: Number of people alive at any given day. The solid black line denotes the mean number of people alive from day 1 until the current day. The empirical and theoretical means are again (see Figure 2) plotted as dotted reference lines. The simulated data are based on $\lambda = 10$ and $\mu = 0.5$.

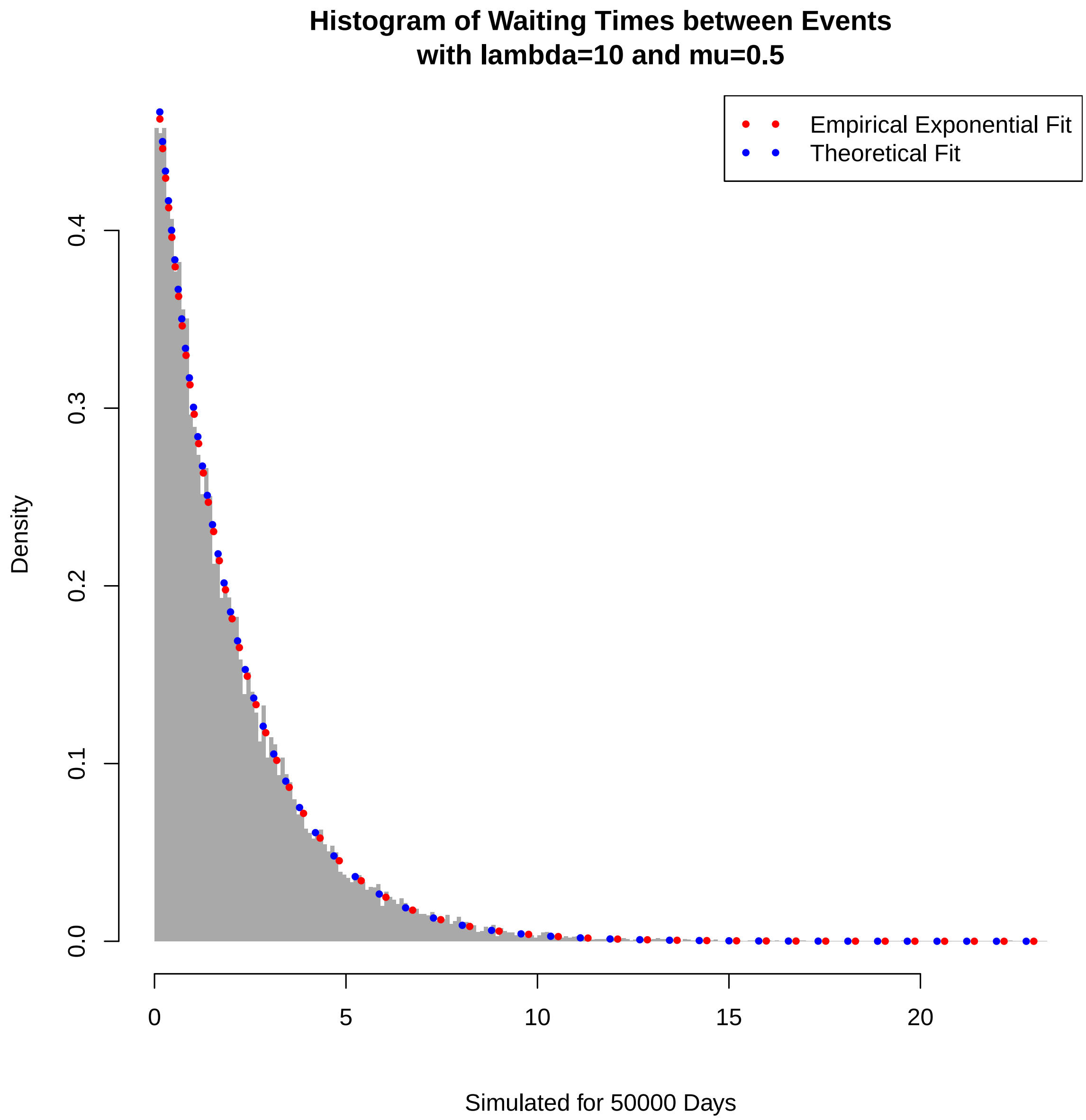


Figure 4: Histogram for the duration between two time points when the (respective) oldest person alive dies. The theoretical fit denotes the density for an exponential distribution parameter $\mu$. The empirical fit depicts a plot of the density for an exponential distribution with $\hat{\mu}$ estimated from the data. The simulated data are based on $\lambda = 10$ and $\mu = 0.5$.

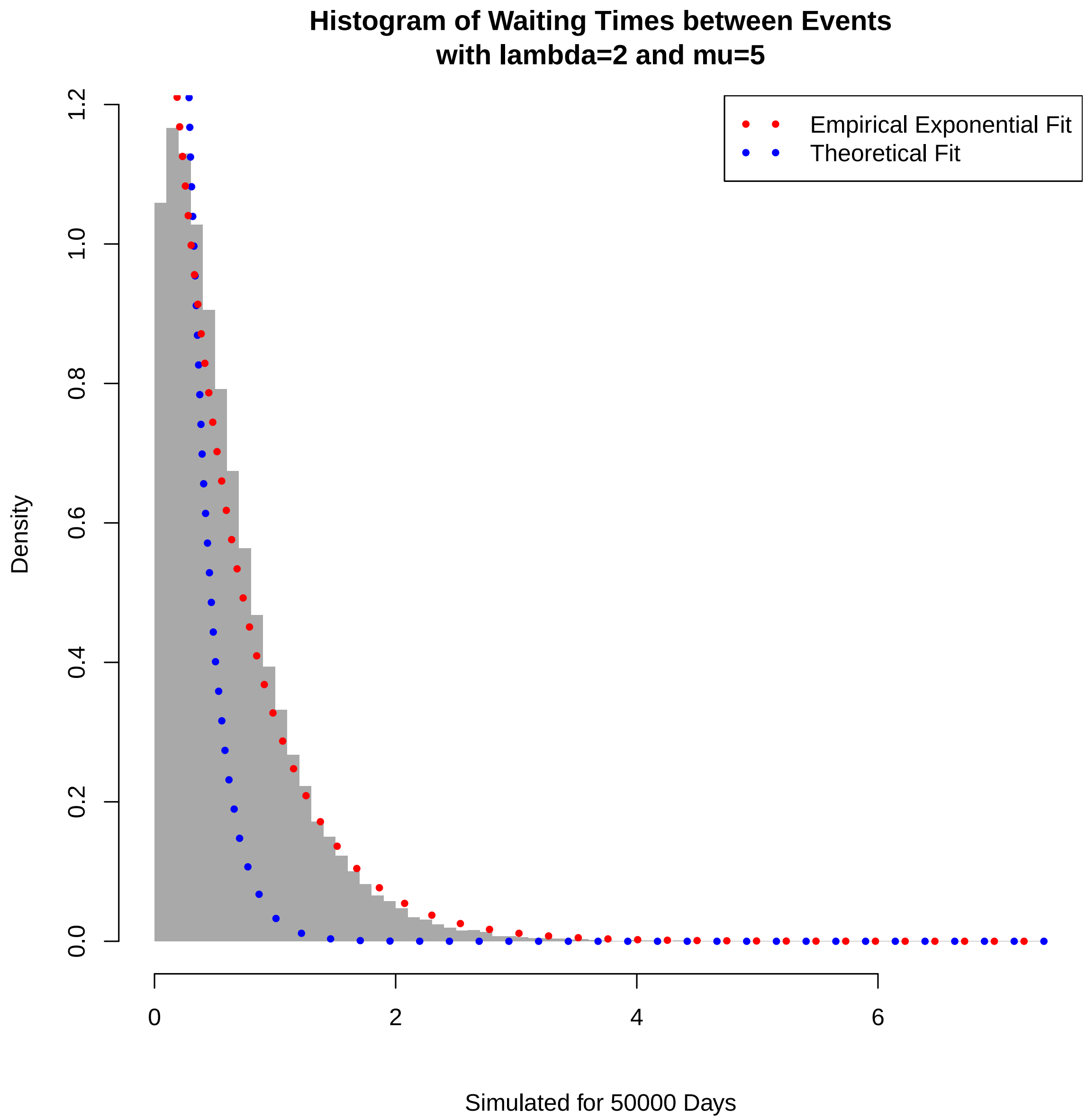


Figure 5: Histogram for the duration between two time points when the (respective) oldest person alive dies. The theoretical fit denotes the density for an exponential distribution parameter $\mu$. The empirical fit depicts a plot of the density for an exponential distribution with $\hat{\mu}$ estimated from the data. The simulated data are based on $\lambda = 10$ and $\mu = 0.5$.